CSE 599S Proof Complexity & Applications

Lecture 18 2 Dec 2020

Sherali-Adams

$\mathcal{P} = \{p_1, \ldots, p_m\}$ $p_0 = 1$

Given $\{p_1 \geq 0, \ldots, p_m \geq 0\}$ derive $p \geq 0$

- $g_0 + \sum_{i \in [m]} g_i \cdot p_i \equiv_I p$

g_i { non-neg sum of $J_{P,N} = \prod_{i \in P} x_i \prod_{i \in N} \bar{x}_i$

conical junta

$\bar{x}_i = 1 - x_i$

Deg d SA pseudo expectation $\tilde{\mathbb{E}}$ for $\mathcal{P}$ on $\mathbb{R}[x_1 \ldots x_n]/I$

multilinear polys with $x_i^2 = x_i$

- linear
- $\tilde{\mathbb{E}}(1) = 1$
- $\tilde{\mathbb{E}}(J_{P,N}(x)) \geq 0$ $\forall P, N$ $|P| + |N| \leq d$
- $\tilde{\mathbb{E}}(J_{P,N}(x) \cdot p_i(x)) \geq 0$ $i \in [m]$ $|P| + |N| \leq d - \deg(p_i)$

$\tilde{\mathbb{E}}(x_i)$

$\mathcal{E}_d^{SA}(\mathcal{P})$ = set of all SA $\tilde{\mathbb{E}}$ of deg d for $\mathcal{P}$

Thm SA derives p in degree d iff $\forall \tilde{\mathbb{E}} \in \mathcal{E}_d^{SA}(\mathcal{P})$ $\tilde{\mathbb{E}}(p) \geq 0$

easy

false if $p = -1$

SOS

$$q_0' + \sum_{i \in [m]} q_i' p_i \equiv_I p$$

q_i' is a SOS

deg d SoS-pseudo-expectation

$\tilde{\mathbb{E}}$ for $\mathcal{P}$

- linear
- $\tilde{\mathbb{E}}(1) = 1$
- $\tilde{\mathbb{E}}(q^2(x)) \geq 0$ $\deg q \leq d/2$
- $\tilde{\mathbb{E}}(\underbrace{q^2(x)p_i(x)}_{\deg \leq d}) \geq 0$ $\deg q \leq \frac{d - \deg p_i}{2}$
- $\mathcal{E}_d^{sos}(\mathcal{P})$ = set of all such $\tilde{\mathbb{E}}$

Thm SA derives p in degree d iff $\forall \tilde{\mathbb{E}} \in \mathcal{E}_d^{sos}(\mathcal{P})$ have $\tilde{\mathbb{E}}(p) \geq 0$ Jeary

---

Finding $\tilde{\mathbb{E}}$: define $x_S = \prod_{i \in S} x_i$

by linearity $\tilde{\mathbb{E}}$ determined by $\tilde{\mathbb{E}}(x_S)$ $|S| \leq d$

variables y_S for each $S \in \binom{[n]}{\leq d}$

$y_\emptyset = 1$

solve for y_S to get $\tilde{\mathbb{E}}$

SA:

$$y_\phi = 1$$

$$J_{P,N}(x) = x_P \cdot \prod_{i \in N} (1 - x_i)$$

$$= x_P - \sum_{i \in N} x_{P \cup \{i\}} + \sum_{i,j \in N} x_{P \cup \{i,j\}}$$

$\qquad y_P \qquad y_{P \cup \{i\}} \quad \cdots$

$\tilde{\mathbb{E}}(J_{P,N}(x)) \geq 0$

$\Downarrow$

$y_P - \sum_i y_{P \cup \{i\}} + \cdots \geq 0$

$J_{P,N}(x) \cdot p_i(x)$

Linear program in Y_S vars

$$A\vec{y} \geq 0$$
$$y_\phi = 1$$

LP with $\binom{n}{\leq d}$ vars

$|P|,|N| \leq d \quad i \in [m]$

$< \binom{n}{\leq d}^2$ rows

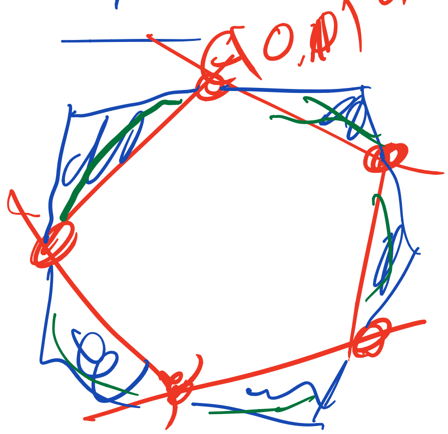
polytime alg for LP

running time $\binom{n}{\leq d}^{O(1)}$

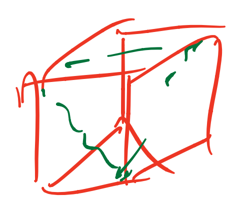
$n^{O(d)}$

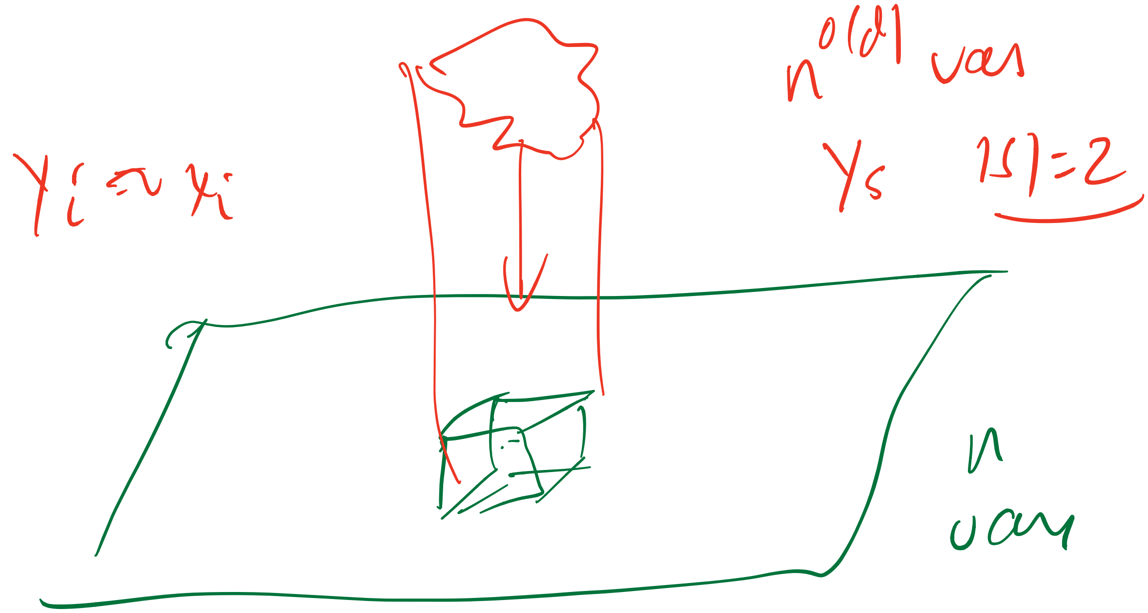
y_i

$\{0,1\}^n$



n vars





SOS solution Y_S

$Y_\emptyset = 1$

$\tilde{\mathbb{E}}(q^2(x)) \geq 0$

$$q(x) = \sum_{|S| \leq d} q_S X_S, \quad q_S \in \mathbb{R}$$

$$0 \leq \tilde{\mathbb{E}}(q^2(x)) = \tilde{\mathbb{E}}\Big(\big(\sum_{|S| \leq d} q_S X_S\big)^2\Big)$$

$$= \tilde{\mathbb{E}}\Big(\sum_{\substack{|S| \leq d \\ |T| \leq d}} q_S \cdot q_T \cdot X_{S \cup T}\Big)$$

$$= \sum_{|S| \leq d} \sum_{|T| \leq d} q_S \cdot q_T \cdot \tilde{\mathbb{E}}(X_{S \cup T})$$

$$= \begin{bmatrix} \quad q_S \quad \end{bmatrix}^{\binom{n}{\le d}} \begin{bmatrix} & & \\ & \cdot & \\ & & \end{bmatrix} \begin{bmatrix} q_T \end{bmatrix} \Big\} \binom{n}{\le d}$$

$M_{\hat{\mathbb{E}}}$

$$(M_{\hat{\mathbb{E}}})_{S,T} = \hat{\mathbb{E}}(X_{S \cup T})$$

$$= q^T \cdot M_{\hat{\mathbb{E}}} \cdot q$$

$M_{\hat{\mathbb{E}}}$ symmetric

psd

$\Longleftrightarrow$ $M_{\mathbb{E}}$ is positive semidefinite (all eigenvalues of real ≥ 0)

" $M_{\hat{\mathbb{E}}} \succeq 0$

$$p_i(x) = \sum_S p_{i,S} \cdot X_S$$

$$0 \leq \hat{\mathbb{E}}(q^2 p_i) = \sum_{S,T} q_S \cdot q_T \cdot \sum_U p_{i,U} \cdot X_{S \cup T \cup U}$$

$$= q^T \cdot M_{\hat{\mathbb{E}}, p_i} \cdot q$$

$$\left(M_{\hat{\tilde{E}}, p_i}\right)_{S,T} = \sum_u p_{i,u}\, \hat{\tilde{E}}(X_{S \cup T \cup u})$$

$(|S \cup T \cup u| \le d)$

$p_{i,u}$ are fixed given ϕ

Fix u $\left(\hat{\tilde{E}}(X_{S \cup T \cup u})\right)_{S,T}$ is a submatrix of $M_{\hat{\tilde{E}}}$ matrix

$M_{\hat{\tilde{E}}, p_i}$ is a fixed linear comb. of submatrices of $M_{\hat{\tilde{E}}}$

need $M_{\hat{\tilde{E}}, p_i} \succeq 0 \ \forall i$

$\exists$ semidefinite program of size $\binom{n}{\le d}$

Solve for $y_S = \mathbb{E}[X_S]$

poly time

Thm Let $\mathcal{P}$ be a set of poly inequalities. There is an alg A st

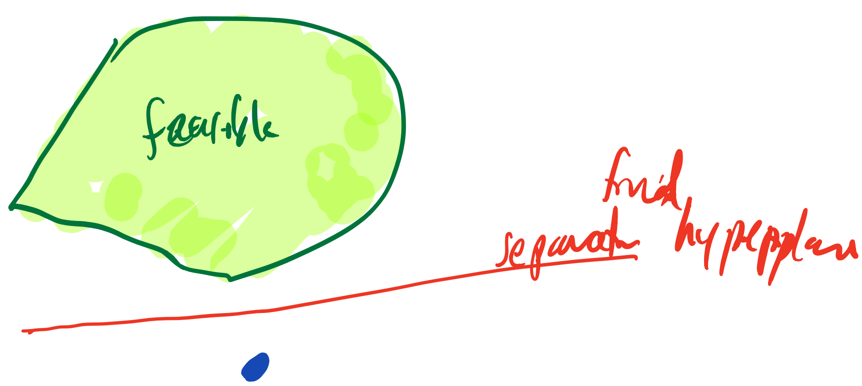
$\forall\, d, \varepsilon$ $\quad A$ runs in time $n^{O(d)} \cdot \mathrm{poly}(\log(1/\varepsilon))$

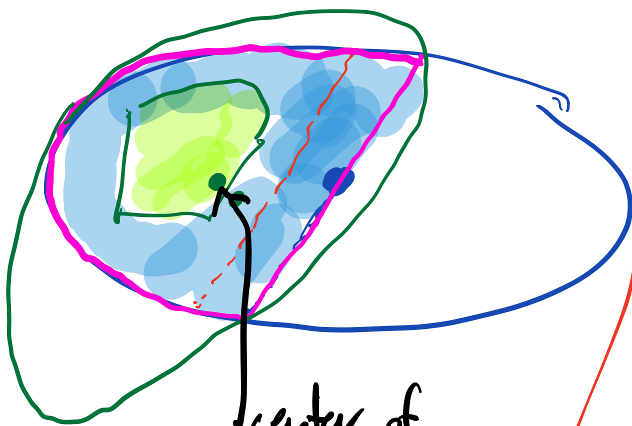
solving $\min \tilde{\mathbb{E}}[q]$

s.t. $\tilde{\mathbb{E}} \in \mathcal{E}_d^{SOS}(\mathcal{P})$

where each constraint is satisfied with additive error $\leq \varepsilon$

Proof Ellipsoid method for convex sets.





Ellipsoid are current guess

start with an outer ellipsoid sphere of radius 1 (contains all sol^{n}) centered at O

center of new ellipse

new guess

new ellipse volume constant factor smaller than original

---

eg. MAX-CUT graph $G=(V,E)$ $V=\{1,\dots,n\}$ weights on edges w_{ij}

weights edges is maximized

max $w^T \cdot x$

$c^T x - k \geq 0$

$\Downarrow$

$OPT \geq k$.

NP-hard

LP gives $\frac{1}{2}$ approx $\geq \frac{1}{2}$ OPT

Goemans-Williamson $\geq (0.878) \cdot OPT$ SDP

(LP (SA of degree $\Omega(n)$ can't get $\geq \frac{1}{2}+\varepsilon$)

GW alg $\equiv$ deg 2 SOS

$x_i \in \{0,1\}$

objective max $\sum_{i,j} w_{ij} (x_i - x_j)^2$ $\{0,1\}$

Unique Games Conjecture (UGC)
Khot 2002

Input: directed graph $G = (V, E)$ $V = \{1, \dots, n\}$

Max-LIN$_p$ value b_{ij} $\in \mathbb{Z}_p$ for all $(i,j) \in E$

Goal: Fix x_i to maximize number fraction of $(i,j) \in E$

s.t. $x_j - x_i = b_{ij} \pmod p$

random guess for x_i $\downarrow$ value $1/p$

UGC: $\forall \varepsilon > 0$ $\exists p$ s.t. it is NP-hard to tell iff Max-LIN$_p$ value $\geq 1-\varepsilon$ or $\leq \varepsilon$

Fact: (Hastad): NP hard to tell for $p = 2$ if $\leq \frac{1}{2} + \varepsilon$ or $\geq 1 - \varepsilon$ $\forall \varepsilon > 0$

new: NP hard to tell if $\leq \varepsilon$ or $\geq \frac{1}{2} - \varepsilon$ for all p

Thm (Raghavendra)

UGC $\Rightarrow$ For any $\varepsilon > 0$ and any constraint satisfaction problem CSP, the MAX-CSP it is NP-hard to obtain a solution with value $\geq K + \varepsilon$ where K is maximum s.t. deg 2 SOS can derive polytime a solution $C^T X - K \geq 0$

eg col, 3SAT, LIN, ..., objective $C^T \cdot X$

universal alg

Also Best alg for MAX-LIN$_p$ known is degree $O(n^{1/3})$ SOS $\tilde{O}(n^{1/3})$ alg

---

Degree lower bounds for SOS

Fact: size $\rightarrow$ degree conversion for SOS

$\{0,1\}$: $S \rightarrow O\left(\sqrt{n \log S} + \deg p\right)^{4}$

Random 3-CNF requires degree $\Omega(n)$ in SOS

$\exists$ come up with a d pseudo-expectation for $d = \Omega(n)$

Just like for PC convenient to convert to random 3XOR

$x_i \in \{0,1\} \longmapsto z_i \in \{1,-1\}$

parity equation

pseudo-expectation constraints on z_i

multilinear

$x_i^2 = x_i \longmapsto z_i^2 = 1$

$$(M_{\tilde{E}})_{S,T} = \tilde{E}(X_{S \cup T})$$

$$\downarrow$$

$\tilde{E}(1) = 1$

$\tilde{E}(z_i^2 p(z)) = \tilde{E}(p)$

$\tilde{E}(q^2 p_i(z)) \geq 0$

$\tilde{E}(q^2(z)) \geq 0$

$$(M_{\tilde{E}})_{S,T} = \tilde{E}(Z_{S \oplus T})$$

deg: same